

Zero-Error Function Computation through a Bidirectional Relay

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Abstract—We consider zero error function computation in a three node wireless network. Nodes A and B observe X and Y respectively, and want to compute a function $f(X, Y)$ with zero error. To achieve this, nodes A and B send messages to a relay node C at rates R_A and R_B respectively. The relay C then broadcasts a message to A and B at rate R_C to help them compute $f(X, Y)$ with zero error. We allow block coding, and study the region of rate-triples (R_A, R_B, R_C) that are feasible. The rate region is characterized in terms of graph coloring of some suitably defined probabilistic graphs. We give single letter inner and outer bounds which meet for some simple examples. We provide a sufficient condition on the joint distribution p_{XY} under which the relay can also compute $f(X, Y)$ if A and B can compute it with zero error.

I. INTRODUCTION

Distributed computation of distributed data is a common problem in a network. Such problems in various flavours have attracted strong research interest in the last decade. Gathering all the data at the nodes where a function needs to be computed is wasteful in most situations. So intermediate nodes also help by doing some processing of the data to reduce the communication load on the links. Such computation frameworks are known as distributed function computation or in-network function computation [1], [2], [3], [4].

We consider the problem of function computation in a wireless network with three nodes as shown in Fig. 1. Nodes A and B have two correlated random variables X and Y respectively. They have infinite i.i.d. realizations of these random variables. They can communicate directly to a relay node C over orthogonal error-free links. The relay node C can broadcast a message to both A and B. A and B receive such broadcasted message without error. Nodes A and B want to compute a function $f(X, Y)$ with zero error for all realizations of (X, Y) with nonzero probability. We allow block coding of arbitrarily large block length n . For each block of n data symbols, we allow two phases of communication. In the first phase, both A and B send individual messages to C over the respective orthogonal links. In the second phase, the relay broadcasts a message to A and B. We study the expected number of bits that need to be sent per computation over the individual links (A,C), (B,C) and over the broadcast link from C to A and B. Since all the nodes transmit once in our protocol, we call this a one-round protocol.

The problem of zero error source coding with receiver side information was first studied by Witsenhausen in [5]. Here,

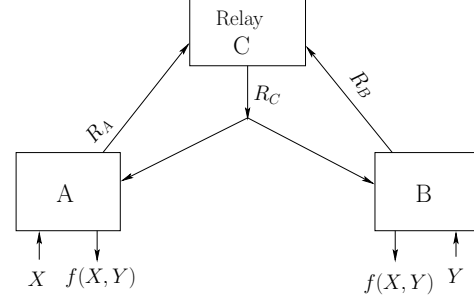


Fig. 1: Function computation in bidirectional relay network

the receiver has side information Y and it wants to recover the source random variable X with zero error. A confusability graph and its AND product graphs were defined, and the minimum rate was characterized in terms of the chromatic number of the n -times AND product graph. The same side information problem was later considered in [6] in the context of variable length coding. The goal was to minimize the expected number of bits per symbol that need to be sent from the transmitter. This quantity was shown to be the limit of the chromatic entropy of the AND product of the confusability graph. This asymptotic rate was later shown [7] to be the complementary graph entropy [8] of the confusability graph. A single letter characterization for complementary graph entropy is still unknown.

In [6], protocols are further considered for the special case of unrestricted inputs, where (X, Y) may take values outside the support of p_{XY} . It was shown that the minimal asymptotic rate is the chromatic entropy of the OR product graph of the confusability graph. It was further shown that this rate can be expressed as the graph entropy [9], [10] of the confusability graph - thus resulting in a single-letter characterization.

Distributed encoding of two correlated sources and joint decoding with zero error is considered in [11]. A Single letter characterization is given for the achievable region under unrestricted inputs, and this gives an inner bound for the original problem under restricted inputs. Another unidirectional network is considered in [12] for distributed computation. Here two distributed sources (with X and Y) encode and send messages to a common relay, which in turn sends a message to a decoder with side-information Z . The decoder wants to compute a function $f(X, Y, Z)$ with zero error. The input is

assumed to be unrestricted. The idea of graph entropy region of a probabilistic graph is introduced, and some single letter inner and outer bounds for the graph entropy region are given.

For our zero-error computation problem depicted in Fig. 1, we provide a characterization of the rate region in terms of graph coloring of some suitably defined graphs. We provide single letter inner and outer bounds for the rate region. A sufficient condition on the joint distribution p_{XY} is identified under which, the relay will also be able to reconstruct $f(X, Y)$ for any scheme where A and B reconstruct it with zero error.

The paper is organized as follows. Section II presents problem formulation and some definitions. In Section III, we define graph entropy region and state our main results. Proof of the results and some examples are given in Section IV. We conclude the paper in Section V.

II. PROBLEM FORMULATION AND SOME DEFINITIONS

A. Problem formulation

Nodes A and B observe X and Y respectively from finite alphabet sets \mathcal{X} and \mathcal{Y} . (X, Y) have a joint distribution $p_{XY}(x, y)$, and their different realizations are i.i.d. In other words, n consecutive realizations (X^n, Y^n) are distributed as $Pr(x^n, y^n) = \prod_{i=1}^n p_{XY}(x_i, y_i)$ for all $x^n = (x_1, x_2, \dots, x_n)$ and $y^n = (y_1, y_2, \dots, y_n)$.

The support set of (X, Y) is defined as

$$S_{XY} = \{(x, y) : p_{XY}(x, y) > 0\}.$$

On observing X^n and Y^n respectively, A and B send messages M_A and M_B using prefix free codes such that $E|M_A| = nR_A$ and $E|M_B| = nR_B$. Here $|\cdot|$ denotes the length of the respective message in bits. C then broadcasts a message M_C with $E|M_C| = nR_C$ to A and B. Each of A and B then decode $f(X_i, Y_i)$; $i = 1, 2, \dots, n$ from the information available to them. A length- n scheme is a quintuple $(\phi_A, \phi_B, \phi_C, \psi_A, \psi_B)$, where

$$\phi_A : \mathcal{X}^n \longrightarrow \{0, 1\}^*, \quad \phi_B : \mathcal{Y}^n \longrightarrow \{0, 1\}^*$$

and

$$\phi_C : \phi_A(\mathcal{X}^n) \times \phi_B(\mathcal{Y}^n) \longrightarrow \{0, 1\}^*$$

are prefix free encoding functions of A, B and C respectively, and

$$\psi_A : \mathcal{X}^n \times \phi_C(\phi_A(\mathcal{X}^n) \times \phi_B(\mathcal{Y}^n)) \longrightarrow \mathcal{Z}^n$$

and

$$\psi_B : \mathcal{Y}^n \times \phi_C(\phi_A(\mathcal{X}^n) \times \phi_B(\mathcal{Y}^n)) \longrightarrow \mathcal{Z}^n$$

are the decoding functions of A and B. Here $\{0, 1\}^*$ denotes the set of all finite length binary sequences. Let $(\psi_A(\cdot))_i$ and $(\psi_B(\cdot))_i$ denote the i -th components of $\psi_A(\cdot)$ and $\psi_B(\cdot)$ respectively. A scheme is called a *zero-error scheme* if for each $X^n \in \mathcal{X}^n, Y^n \in \mathcal{Y}^n$, and $i = 1, 2, \dots, n$,

$$(\psi_A(X^n, \phi_C(\phi_A(X^n), \phi_B(Y^n))))_i = f(X_i, Y_i)$$

and

$$(\psi_B(Y^n, \phi_C(\phi_A(X^n), \phi_B(Y^n))))_i = f(X_i, Y_i)$$

if $(X_i, Y_i) \in S_{XY}$.

The rate triplet (R_A, R_B, R_C) of a scheme is defined as

$$R_A = \frac{1}{n} \sum_{x^n} Pr(x^n) |\phi_A(x^n)|$$

$$R_B = \frac{1}{n} \sum_{y^n} Pr(x^n) |\phi_B(y^n)|$$

$$R_C = \frac{1}{n} \sum_{(x^n, y^n)} Pr(x^n, y^n) |\phi_C(\phi_A(x^n), \phi_B(y^n))|.$$

A rate-triple is said to be achievable if there is a zero-error scheme of some length with that rate-triple. The rate-region $\mathcal{R}(f, X, Y)$ is the closure of the set of achievable rate-triples.

B. Graph theoretic definitions

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is called an independent set if no two vertices in S are adjacent in G . The n -fold OR product of G , denoted by $G^{\vee n}$, is defined by $V(G^{\vee n}) = (V(G))^n$ and $E(G^{\vee n}) = \{(v^n, v'^n) : (v_i, v'_i) \in E(G) \text{ for some } i\}$.

For a graph G and a random variable X taking values in $V(G)$, (G, X) represents a *probabilistic graph*. Chromatic entropy [6] of (G, X) is defined as

$$H_X(G, X) = \min\{H[c(X)] : c \text{ is a coloring of } G\}.$$

Let W be distributed over the power set $2^{\mathcal{X}}$. The graph entropy of the probabilistic graph (G, X) is defined as

$$H_G(X) = \min_{X \in W \in \Gamma(G)} I(W; X), \quad (1)$$

where $\Gamma(G)$ is the set of all independent sets of G . Here the minimum is taken over all conditional distribution $p_{W|X}$ which is non-zero only for $X \in W$. The following interesting result was shown in [6].

$$\lim_{n \rightarrow \infty} \frac{1}{n} H_X(G^{\vee n}, X^n) = H_G(X). \quad (2)$$

We now define some graphs suitable for addressing our problem. For a function $f(x, y)$ defined over $\mathcal{X} \times \mathcal{Y}$, we define a graph called f -modified rook's graph. A rook's graph G_{XY} over $\mathcal{X} \times \mathcal{Y}$ is defined by the vertex set $\mathcal{X} \times \mathcal{Y}$ and edge set $\{(x, y), (x', y') : x = x' \text{ or } y = y', \text{ but } (x, y) \neq (x', y')\}$.

Definition 1 For a function $f(x, y)$ the f -modified rook's graph G_{XY}^f has its vertex set S_{XY} , and two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if they are adjacent in the rook's graph G_{XY} and $f(x_1, y_1) \neq f(x_2, y_2)$.

For example, let us consider $X, Y \in \{0, 1, 2, 3, 4\}$ with distribution

$$p(x, y) = \begin{cases} \frac{1}{10} & \text{if } y = x \text{ or } y = x + 1 \pmod{5} \\ 0 & \text{otherwise} \end{cases},$$

and the equality function

$$f(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases} \quad (3)$$

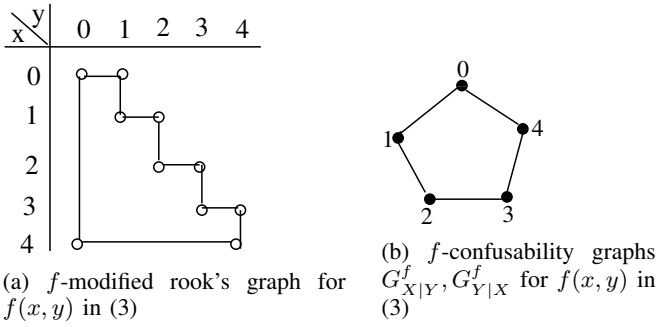


Fig. 2: f -modified rook's graph and f -confusability graph

The f -modified rook's graph for this function is shown in Fig. 2a.

f -confusability graph $G_{X|Y}^f$ of X, Y and f was defined in [12]. Its vertex set is \mathcal{X} , and two vertices x and x' are adjacent if and only if $\exists y \in \mathcal{Y}$ such that $f(x, y) \neq f(x', y)$ and $(x, y), (x', y) \in S_{XY}$. $G_{Y|X}^f$ is defined similarly. For the function defined in (3), $G_{X|Y}^f$ and $G_{Y|X}^f$ are the same graph which is shown in Fig. 2b.

III. SUMMARY OF RESULTS

A. Characterization of $\mathcal{R}(f, X, Y)$

We first define the chromatic entropy region of a f -modified rooks graph. If c_1 and c_2 are two maps of \mathcal{X} and \mathcal{Y} respectively, then $c_1 \times c_2$ denotes the map given by $(c_1 \times c_2)(x, y) = (c_1(x), c_2(y))$.

Recall that S_{XY} is the vertex set of G_{XY}^f . A triplet (c_A, c_B, c_C) of functions defined over $\mathcal{X}, \mathcal{Y}, S_{XY}$ respectively is called a *color cover* for G_{XY}^f if

- i) $c_A \times c_B$ and c_C are colorings of G_{XY}^f .
- ii) $c_A \times c_B$ is a refinement of c_C in S_{XY} , i.e., \exists a mapping $\theta : (c_A \times c_B)(S_{XY}) \rightarrow c_C(S_{XY})$ such that $\theta \circ (c_A \times c_B) = c_C$.

Chromatic entropy region $R_X(G_{XY}^f, X, Y)$ of G_{XY}^f is defined as

$$R_X(G_{XY}^f, X, Y) \triangleq \bigcup_{(c_A, c_B, c_C)} \{(b_A, b_B, b_C) : b_A \geq H(c_A(X)), b_B \geq H(c_B(Y)), b_C \geq H(c_C(X, Y))\},$$

where the union is taken over all color covers of G_{XY}^f . Motivated by the result (2), we define the *graph entropy region* as

$$HR_{G_{XY}^f}(X, Y) \triangleq \bigcup_n \frac{1}{n} R_X((G_{XY}^f)^{\vee n}, X^n, Y^n).$$

Theorem 1 (i) $\mathcal{R}(f, X, Y) = HR_{G_{XY}^f}(X, Y)$.

(ii) Let

$$\begin{aligned} \mathcal{R}_{I1} &\triangleq \{(R_A, R_B, R_C) : R_A \geq H(X), R_B \geq H(Y), \\ &\quad R_C \geq H_{G_{XY}^f}(X, Y)\} \\ \mathcal{R}_{I2} &\triangleq \{(R_A, R_B, R_C) : R_A \geq H_{G_{X|Y}^f}(X), \\ &\quad R_B \geq H_{G_{Y|X}^f}(Y), \\ &\quad R_C \geq H_{G_{X|Y}^f}(X) + H_{G_{Y|X}^f}(Y)\}. \end{aligned}$$

Let \mathcal{R}_I be the convex hull of $\mathcal{R}_{I1} \cup \mathcal{R}_{I2}$. Then $\mathcal{R}_I \subseteq \mathcal{R}(f, X, Y)$.

(iii) Let

$$\begin{aligned} \mathcal{R}_O &\triangleq \{(R_A, R_B, R_C) : R_A \geq H_{G_{X|Y}^f}(X), \\ &\quad R_B \geq H_{G_{Y|X}^f}(Y), R_C \geq H_{G_{XY}^f}(X, Y)\}. \end{aligned}$$

Then $\mathcal{R}(f, X, Y) \subseteq \mathcal{R}_O$.

(iv) If $G_{Y|X}^f$ and $G_{X|Y}^f$ are complete graphs, then $\mathcal{R}_I = \mathcal{R}_O$.

Theorem 2 If $p(x, y) > 0 \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$, then for any zero-error scheme the relay can compute $f(x, y)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

IV. PROOFS OF THE RESULTS

To prove Theorem 1, we first present some lemmas.

Lemma 1 For $n = 1$, and given the encoding functions ϕ_A, ϕ_B, ϕ_C , the nodes A and B can recover $f(X, Y)$ with zero error if and only if $\phi_C \circ (\phi_A \times \phi_B)$ is a coloring of G_{XY}^f .

Proof: Let $E(G_{XY}^f)$ denote the set of edges of G_{XY}^f . Note that

$$\begin{aligned} E(G_{XY}^f) &= \{((x, y), (x, y')) \in S_{XY} | f(x, y) \neq f(x, y')\} \\ &\cup \{((x, y), (x', y)) \in S_{XY} | f(x, y) \neq f(x', y)\} \end{aligned} \quad (4)$$

Note that each edge is of the form $((x, y), (x, y'))$ or $((x, y), (x', y))$.

A and B can recover $f(X, Y)$ with zero error \Leftrightarrow (i) for any $(x, y), (x, y') \in S_{XY}$ with $f(x, y) \neq f(x, y')$, $\phi_C(\phi_A(x), \phi_B(y)) \neq \phi_C(\phi_A(x), \phi_B(y'))$ and (ii) for any $(x, y), (x', y) \in S_{XY}$ with $f(x, y) \neq f(x', y)$, $\phi_C(\phi_A(x), \phi_B(y)) \neq \phi_C(\phi_A(x'), \phi_B(y)) \Leftrightarrow$ for any $((x, y), (x', y')) \in E(G_{XY}^f)$, $\phi_C(\phi_A(x), \phi_B(y)) \neq \phi_C(\phi_A(x'), \phi_B(y')) \Leftrightarrow \phi_C \circ (\phi_A \times \phi_B)$ is a coloring of G_{XY}^f . ■

Clearly, whenever $\phi_C \circ (\phi_A \times \phi_B)$ is a coloring of G_{XY}^f , $\phi_A \times \phi_B$ is also a coloring of G_{XY}^f . The following lemma gives a necessary and sufficient condition for (ϕ_A, ϕ_B) to be a coloring of G_{XY}^f .

Lemma 2 (ϕ_A, ϕ_B) is a coloring of G_{XY}^f if and only if ϕ_A is a coloring of $G_{X|Y}^f$ and ϕ_B is a coloring of $G_{Y|X}^f$.

Proof: ϕ_A and ϕ_B are colorings of $G_{X|Y}^f$ and $G_{Y|X}^f$ respectively \Leftrightarrow (i) any $(x, y), (x, y') \in S_{XY}$ with $f(x, y) \neq f(x, y')$, satisfies $(\phi_A(x), \phi_B(y)) \neq (\phi_A(x), \phi_B(y'))$ and (ii) any $(x, y), (x', y) \in S_{XY}$ with $f(x, y) \neq f(x', y)$ satisfies $(\phi_A(x), \phi_B(y)) \neq (\phi_A(x'), \phi_B(y)) \Leftrightarrow \phi_A \times \phi_B$ is a coloring of G_{XY}^f . We used (4) in the last equivalence. ■

We now extend the above lemmas for n -length schemes.

Lemma 3 For any n , and given the encoding functions ϕ_A, ϕ_B, ϕ_C , the nodes A and B can recover $f(X_i, Y_i)$; $i = 1, 2, \dots, n$ with zero error if and only if $\phi_C \circ (\phi_A \times \phi_B)$ is a coloring of $(G_{XY}^f)^{\vee n}$.

Proof: Consider some $x^n \in \mathcal{X}^n$ and $y^n, y'^n \in \mathcal{Y}^n$ such that for some i , $p(x_i, y_i) \cdot p(x_i, y'_i) > 0$ and $f(x_i, y_i) \neq f(x_i, y'_i)$. For pairs (x^n, y^n) and (x^n, y'^n) node A should receive different data from the relay for zero error computation in the unrestricted setup. Similarly node B should receive different data for pairs (x^n, y^n) and (x'^n, y^n) if for some i , $f(x_i, y_i) \neq f(x'_i, y_i)$ and $(x_i, y_i), (x'_i, y_i) \in S_{XY}$. This shows that zero-error computation is possible if and only if $\phi_C \circ (\phi_A \times \phi_B)$ is a coloring of $(G_{XY}^f)^{\vee n}$. ■

Lemma 4 $\phi_A \times \phi_B$ is a coloring of $(G_{XY}^f)^{\vee n}$ if and only if ϕ_A is a coloring of $(G_{X|Y}^f)^{\vee n}$ and ϕ_B is a coloring of $(G_{Y|X}^f)^{\vee n}$.

Proof: If ϕ_A is not a coloring of $(G_{X|Y}^f)^{\vee n}$, then $\exists x^n, x'^n \in \mathcal{X}^n$ and $y^n \in \mathcal{Y}^n$ with $(x_i, y_i), (x'_i, y_i) \in S_{XY}, f(x_i, y_i) \neq f(x'_i, y_i)$ for some i such that $\phi_A(x^n) = \phi_A(x'^n)$. Then $((x^n, y^n), (x'^n, y^n)) \in E((G_{XY}^f)^{\vee n})$, and $(\phi_A(x^n), \phi_B(y^n)) = (\phi_A(x'^n), \phi_B(y^n))$. So $\phi_A \times \phi_B$ is not a coloring of $(G_{XY}^f)^{\vee n}$. Similarly, if ϕ_B is not a coloring of $(G_{Y|X}^f)^{\vee n}$, then $\phi_A \times \phi_B$ is not a coloring of $(G_{XY}^f)^{\vee n}$. The opposite implication also follows along similar lines using (4). ■

Lemma 5 For a given n , the rate-triple of any length- n zero-error scheme satisfies

$$\begin{aligned} R_A &\geq \frac{1}{n} H_X \left((G_{X|Y}^f)^{\vee n}, X^n \right) \\ R_B &\geq \frac{1}{n} H_X \left((G_{Y|X}^f)^{\vee n}, Y^n \right) \\ R_C &\geq \frac{1}{n} H_X \left((G_{XY}^f)^{\vee n}, (X^n, Y^n) \right). \end{aligned}$$

Proof: This follows from the definition of chromatic entropy, Lemmas 3 and 4. ■

A. Proof of Theorem 1

Proof: Proof of Part (i): Lemma 3 implies that for encoding functions ϕ_A, ϕ_B, ϕ_C of any zero-error scheme, $\phi_A, \phi_B, \phi_C \circ (\phi_A \times \phi_B)$ is a color cover for $(G_{XY}^f)^{\vee n}$. Similarly, for any color cover (c_A, c_B, c_C) of $(G_{XY}^f)^{\vee n}$, let ϕ_A, ϕ_B be any prefix-free encoding functions of c_A and c_B respectively. Since $c_A \times c_B$ is a refinement of c_C , there

exists a mapping θ_C such that $c_C = \theta_C \circ (c_A \times c_B)$. Taking ϕ_C as any prefix-free encoding of c_C yields a scheme with encoding functions (ϕ_A, ϕ_B, ϕ_C) . Thus the result follows from the definition of graph entropy region $HR_{G_{XY}^f}(X, Y)$.

Proof of Part (ii): Let us consider a zero-error scheme in which nodes A and B communicate X^n and Y^n to the relay node C. On knowing X^n and Y^n , the relay broadcasts a minimum entropy coloring of $(G_{XY}^f)^{\vee n}$ to A and B. Using an optimum prefix free code for each communication, the achieved rates satisfy

$$\begin{aligned} R_A &\leq \frac{1}{n} H(X^n) + \frac{1}{n} \\ R_B &\leq \frac{1}{n} H(Y^n) + \frac{1}{n} \\ \text{and} \\ R_C &\leq \frac{1}{n} H_X \left((G_{XY}^f)^{\vee n}, (X^n, Y^n) \right) + \frac{1}{n}. \end{aligned}$$

Taking $n \rightarrow \infty$, this proves that any point in \mathcal{R}_{I1} is achievable.

Now let us consider another scheme. Node A and B send a coloring of $(G_{X|Y}^f)^{\vee n}$ and $(G_{Y|X}^f)^{\vee n}$ respectively to C. C then broadcasts both these colors to A and B. Again, using optimum prefix free zero error codes, the rates achieved satisfy

$$\begin{aligned} R_A &\leq \frac{1}{n} H_X \left((G_{X|Y}^f)^{\vee n}, X^n \right) + \frac{1}{n} \\ R_B &\leq \frac{1}{n} H_X \left((G_{Y|X}^f)^{\vee n}, Y^n \right) + \frac{1}{n} \\ \text{and} \end{aligned}$$

$$R_C \leq \frac{1}{n} \left(H_X((G_{X|Y}^f)^{\vee n}, X^n) + H_X((G_{Y|X}^f)^{\vee n}, Y^n) + 2 \right).$$

Thus any point in \mathcal{R}_{I2} is achievable. Thus any point in the convex hull \mathcal{R}_I of $\mathcal{R}_{I1} \cup \mathcal{R}_{I2}$ is also achievable by time-sharing.

Proof of Part (iii): As $n \rightarrow \infty$, Lemma 5 and (2) give the outer bound \mathcal{R}_O .

Proof of Part (iv): When G is a complete graph, the only independent sets are the singletons of $V(G)$. So for (G, X) with such a graph, the only choice of W in (1) is $W = \{X\}$ and

$$H_G(X) = I(W; X) = H(X).$$

Thus if $G_{Y|X}^f$ and $G_{X|Y}^f$ are complete graphs, then $H_{G_{X|Y}^f}(X) = H(X)$ and $H_{G_{Y|X}^f}(Y) = H(Y)$. Then $\mathcal{R}_{I1} = \mathcal{R}_O$, and thus the result follows. ■

In the following, we provide an example of a function for which $\mathcal{R}_I = \mathcal{R}_O$.

Example 1 Let X and Y be uniformly distributed over $\{0, 1, 2\}$ and let $S_{XY} = \mathcal{X} \times \mathcal{Y}$. Nodes A and B want to compute $\min(X, Y)$. For this example, the f -confusability graphs $G_{X|Y}^f$ and $G_{Y|X}^f$ are complete graphs with vertex set $\{0, 1, 2\}$. So here we get $\mathcal{R}_I = \mathcal{R}_{I1} = \mathcal{R}_O$.

B. Proof of Theorem 2

Proof: Let us consider any n . With abuse of notation, we denote the messages sent by the nodes A, B, and C by ϕ_A, ϕ_B, ϕ_C respectively. In the following, we omit the arguments, and denote $f(X^n, Y^n)$ by simply f . Since the function is computed with zero error at nodes A and B, we have $H(f|\phi_C, X^n) = 0$ and $H(f|\phi_C, Y^n) = 0$. We want to show that $H(f|\phi_A, \phi_B) = 0$. We prove this by contradiction. Let us assume that $H(f|\phi_A, \phi_B) > 0$. Then $\exists (x^n, y^n)$ and (x'^n, y'^n) such that

$$Pr(X = x^n, Y = y^n, \phi_A = k_1, \phi_B = k_2) > 0 \quad (5)$$

$$Pr(X = x'^n, Y = y'^n, \phi_A = k_1, \phi_B = k_2) > 0 \quad (6)$$

$$\text{and } f(x_i, y_i) \neq f(x'_i, y'_i) \text{ for some } i. \quad (7)$$

We consider two cases.

Case 1: $x_i = x'_i = x$. Since we have $Pr(X^n = x^n, \phi_A = k_1) > 0$ (using (5)), we get $Pr(X^n = x^n, Y^n = y'^n, \phi_A = k_1) = Pr(X^n = x^n, \phi_A = k_1)Pr(Y^n = y'^n|X^n = x^n) > 0$. So we get

$$Pr(X^n = x^n, Y = y'^n, \phi_A = k_1, \phi_B = k_2) > 0 \quad (8)$$

as $\phi_B(y'^n) = k_2$.

Taking $k_0 = \phi_C(k_1, k_2)$, (5), (8) imply that $Pr(X^n = x^n, Y^n = y^n, \phi_C = k_0) > 0$ and $Pr(X^n = x^n, Y^n = y'^n, \phi_C = k_0) > 0$.

This, together with (7) gives $H(f|\phi_C, X^n) > 0$. Thus A can not recover $f(X, Y)$ with zero error - a contradiction.

Case 2: $x_i \neq x'_i$ and $y_i \neq y'_i$. Using (7), we get either $f(x_i, y'_i) \neq f(x_i, y_i)$ or $f(x_i, y'_i) \neq f(x'_i, y'_i)$. W.l.o.g, let us assume $f(x_i, y'_i) \neq f(x_i, y_i)$. Then by combining (8) and (5), and using the fact that $f(x_i, y'_i) \neq f(x_i, y_i)$, we get $H(f|X^n, \phi_C) \neq 0$. Thus A can not recover $f(X^n, Y^n)$ with zero error - a contradiction. This completes the proof of the theorem. ■

Theorem 2 does not hold if $S_{XY} \neq \mathcal{X} \times \mathcal{Y}$. In the following, we consider a simple example to demonstrate this. Here nodes A and B recover the function with zero error, but the relay can not reconstruct the function.

Example 2 Consider $X, Y \in \{1, 2, 3\}$

$$p(x, y) = \begin{cases} \frac{1}{6} & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y. \end{cases}$$

Let ϕ_A, ϕ_B and ϕ_C be as follows.

$$\phi_A = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi_B = \begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi_C = \begin{cases} 1 & \text{if } \phi_A = \phi_B \\ 0 & \text{otherwise.} \end{cases}$$

In this example the relay cannot determine the function value in all the cases. When $\phi_A = \phi_B = 0$, the function value can be both 0 and 1. So here $H(f|\phi_A, \phi_B) > 0$.

V. CONCLUSION

We provided a characterization of the rate region for our function computation problem in terms of graph coloring and established single letter inner and outer bounds of the rate region. A sufficient condition on the f -confusability graphs is identified under which these inner and outer bounds coincide. We also showed that if p_{XY} is non-zero for all pairs of values, then the relay can compute the function if both A and B can compute it. We addressed the problem only for one-round protocols. Investigating the problem under multi-round protocols is an interesting direction of future work.

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